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Towards a causal subquantum theory—a contextual C^*-algebraic approach

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Abstract. Quantum mechanics is enlarged into a contextual subquantum model, with the purpose of describing causally individual quantum systems. This model is based on the spectra of the quantum-mechanical observables. A C^*-algebraic generalization of this model is presented. Various physically interesting aspects of the constructed subquantum model are discussed.

1. Introduction

For the majority of physicists the history of attempts to construct a subquantum theory (more often called hidden variables (HV) theory) terminates at no-go theorems, which historically began with the famous von Neumann theorem [1], later proved physically irrelevant by Bell [2]. Actually, a result of lasting value is that of Bell and Gleason [2, 3] (cf also Kochen and Specker [4]), which has established the fact that one cannot ascribe to an individual quantum system simultaneous unique values of all quantum observables (except if the state space is two dimensional).

Namely, quantum mechanics predicts only mean values of observables $\hat{a}$, and they follow uniquely from the quantum states ρ: $\langle\hat{a}\rangle = \mathrm{Tr}\,\rho\hat{a}$. One would naively expect that this state of affairs is *mutatis mutandis* valid on the subquantum level: the average value $\langle\hat{a}\rangle$ should be substituted by a definite value $\hat{a}(\omega)$ in each subquantum state ω, associated with an individual quantum system. Unfortunately, it is not possible to realize this expectation due to the forementioned Gleason–Bell result.

However, a number of well known successfully realized HV theories, such as those constructed by de Broglie [5], Bohm [6], Bohm–Bub [7] and Wiener–Siegal [8], and, finally, the axiomatic theory by Gudder [9], show that there exists a large family of HV theories not forbidden by the Gleason–Bell theorem. The secret of these models lies in the fact that they are all based on the new concept of *context*, the idea for which was introduced first by Bell [2, 10].

Bell suggested that the information which observables are measured simultaneously with $\hat{a}$ is decisively relevant for a consistent subquantum description. In other words, it is essential to take into account the *measurement context* M, i.e. the set of operators to which $\hat{a}$ belongs. More precisely, if besides $\hat{a}$ and ω, M is also given, only then is there no theoretical obstacle that the individual system has a definite value $\hat{a}(\omega)_M$.

It should be noted that such a theory permits the possibility that in another measurement context M', to which $\hat{a}$ also belongs, the measured observable $\hat{a}$ may have a

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different value $\hat{a}(\omega)_M$, for the same individual system in the same subquantum state ω. Moreover, *contextuality* actually consists in the fact that there necessarily exist triplets $(\hat{a}, M, \omega)$ and $(\hat{a}, M', \omega)$ such that $\hat{a}(\omega)_M \neq \hat{a}(\omega)_{M'}$. Thus, contextuality is indispensable. (A natural way of treating contextuality is pointed out in part (*b*) of the discussion (see section 5).)

Shimony [11] gave a classification of contexts into *algebraic* and *environmental* ones. It is noteworthy that the authors of the mentioned successful HV theories (with the exception of Gudder) had not articulated contextuality in their theories (cf discussion, part (*d*)). The notion of contextuality is also underestimated in the otherwise competent summary of HV theories presented by Belinfante [12].

For our approach Gudder's algebraic contextual HV theory was the most inspiring proof for the existence of such theories. Unfortunately, it did not find an appropriate place in Belinfante's book, which is probably due to its insufficient connection with the rest of the mentioned theories.

Gudder constructed a subquantum model in which contexts play a decisive role, and they are defined as maximal Boolean sub-σ-algebras B of the lattice $P(H)$ of projectors, where H is the Hilbert space of the system. The contexts are the images of the spectral measures of complete observables. Gudder's HV space Ω has the property that one can complete each pure quantum state m by a hidden variable $\omega \in \Omega$, so that the pair (m, ω) 'knows' the answer to each quantum question belonging to a given context B. In more detail, for each context B and for each question (projector) $\hat{p} \in B$ the answer is expressed in terms of a quantity $H_B(m, \omega)(\hat{p})$, which can take the value 1 or 0—the physical system in the subquantum state (m, ω) has or has not the property $\hat{p}$ in the context B, respectively. It must be pointed out that the answer to the same question depends on the context to which it belongs.

We have generalized [13] Gudder's model by assuming that the above answer H_B may not exist on a set of points of zero measure in the space Ω. We have thus obtained an HV theory which contains, besides the original Gudder model, a new symplectic HV theory, as well as its special cases: the known HV theory of Bohm and Bub [7] and that of Wiener and Siegel [8].

Gudder's model was also our initial inspiration in this article. Here we try to eliminate three undesirable ideas suggested by Gudder's work. First, his theory, like all others, starts with the assumption that the pure quantum state m has to be completed by hidden variables ω in order to obtain a subquantum state (m, ω) of the individual quantum system. Second, Gudder restricts his theory to projectors in the framework of quantum logic. Third, Gudder's presentation conveys the impression that his notion of context B (though he does not call it 'context') is the only possibility.

Instead of projectors, we start by taking *quantum observables* as our basic entities, and for the introduction of contexts we utilize suitable algebras of operators. We construct the subquantum states ω out of the points of the spectra of the observables in a way independent of quantum pure states. As to the quantum mixed and pure states, they are now context-independent distributions over subquantum states.

If one eliminates m from the subquantum state and if one takes into account all observables (and not just the projectors), then Gudder's notion of context amounts to commutative von Neumann subalgebras of the algebra $L(H)$ of all bounded linear operators in H.

The above line of investigation will be dealt with in a separate article. Here we confine ourselves to pointing out some of its basic features and disadvantages.

Every commutative von Neumann algebra N (in a separable Hilbert space H) is generated by a single Hermitian element $\hat{a} \in N$ [14]. The algebra N is naturally isomorphic to the algebra of complex measurable essentially bounded functions on the spectrum $\sigma(\hat{a})$. The projectors in N are precisely the spectral projectors of $\hat{a}$, and they form a Boolean σ-algebra.

As to shortcomings, the subquantum space Ω necessarily has the unfortunate topological property that it is extremely disconnected. As a consequence, in such a subquantum space it is not possible to introduce local coordinates and, hence, the dynamical law cannot be formulated in terms of a Hamiltonian approach, in contrast to [13].

In this article the notion of context is varied. In a procedure of generalization a general concept of context is reached as an *abstract* C^*-commutative subalgebra, which, as a special case, becomes the mentioned von Neumann algebra (see part (*a*) of the discussion).

The starting point of this article is to take the Cartesian product of the spectra of all observables as the subquantum space Ω. Then the observables $\hat{a}$ and, for example, $2\hat{a}$ may have mutually completely independent values in the same subquantum state, which is physically unacceptable. Thus the question arises of which functional connections between quantum observables should be preserved on the subquantum level. More precisely, there is a known quantum mechanical theorem that says that if an observable $\hat{b}$ has a sharp value b in a quantum state (of course b necessarily belongs to the point spectrum of $\hat{b}$), and there exists another observable $\hat{a}$ that is a function of $\hat{b}$: $\hat{a}=f(\hat{b})$, then in the same state the observable $\hat{a}$ has the sharp value $f(b)$. It might seem a natural assumption that the same relation holds between all values (belonging either to the point or to the continuous spectrum) of observables on the subquantum level: if an individual system has the value b of $\hat{b}$ and one has $\hat{a}=f(\hat{b})$, then the same individual system in the same subquantum state ω has the value $f(b)$ of the observable $\hat{a}$, where f is a continuous function (reasons for this choice see below).

Unfortunately, it has turned out that *a subquantum theory satisfying this requirement for any continuous f is not possible*, as follows from the theorem of Gleason (see the remark in appendix 1). Hence, we must renounce the above context-independent assumption.

We are guided by the following intuitive reasoning in keeping the continuous functional connections that should be valid also on the subquantum level in a contextual model:

(*a*) It seems reasonable to assume that each point of the spectrum $\sigma(\hat{a})$ (discrete or continuous) has an objective, ontological meaning as a hidden property of the individual quantum system.

(*b*) Further, one should assume that the closeness of two points in the continuous part of the spectrum also has an objective meaning in the following sense: if two points are close in $\sigma(\hat{a})$, then they are also close in the discrete spectrum of each approximating observable (see the discussion, part (*a*)).

The usual approximating observable has the spectral form [1]

$$\sum \lambda_n \hat{E}^{\hat{a}}(a_n, a_{n+1}]$$

where the points a_n break up the real axis, λ_n belongs to the nth interval, and $\hat{E}^{\hat{a}}$ is the spectral measure of $\hat{a}$. These non-continuous functions of $\hat{a}$ violate requirement (*b*), but this can be easily corrected, and we do it in the sequel.

As it was stated above, for the continuous spectra, approximating observables are indispensable, and they are expressed in terms of spectral projectors. Since these are not continuous functions, we connect 0 and 1 continuously in the intervals $(a_n-\varepsilon, a_n+\varepsilon)$. In this way we obtain counterparts of projectors which are continuous functions, and which are empirically indistinguishable (due to the sufficient smallness of ε) from $\hat{E}^{\hat{a}}(a_n, a_{n+1}]$.

If an observable $\hat{a}$ is given, then all the operators of the form

$$\hat{b}=f(\hat{a})$$

where $f:\sigma(\hat{a})\to\mathbb{C}$ is a continuous function, constitute a *commutative* C^* *algebra* A, which can be characterized either as the minimal C^* algebra containing $\hat{a}$, or the one generated by $\hat{a}$.

In the next section we start with the construction of a contextual subquantum model in which each context M is defined as the (commutative) C^* algebra A generated by some observable $\hat{a}$.

Further on, we present a general C^*-algebraic subquantum model. In this general framework the relation between the two special cases of the definition of the context M (von Neumann algebras and minimal C^* algebras) is discussed (see the discussion, part (a)).

2. Simple contextual subquantum models and the spectral realization

Let H be the Hilbert state space of a given quantum system, $L(H)$ the algebra of bounded linear operators in H, and let $O(H)$ denote the real linear space of Hermitian elements (bounded quantum observables) in $L(H)$.

Throughout this paper we deal with subquantum models in which the following *four basic concepts* are incorporated:

(i) *The concept of contextual causality.* A subquantum space Ω is introduced, the elements of which (the subquantum states) correspond to the following specifications: if a subquantum state $\omega\in\Omega$ of an individual system is given, then the value $\hat{a}(\omega)_M\in\sigma(\hat{a})$ of any quantum observable $\hat{a}\in M$ in ω is defined also if a context M is specified.

(ii) *The ignorance interpretation of quantum probabilities.* For each quantum state ρ (statistical operator in H) a Kolmogorov probability measure $(\Omega, \mathbf{P}, \mu_\rho)$ is given such that

$$\mathrm{Tr}(\rho\hat{a})=\int_\Omega \hat{a}(\omega)_M\,\mathrm{d}\mu_\rho(\omega) \tag{2.1}$$

for each $\hat{a}\in 0(H)$ and each context M containing $\hat{a}$.

Remark. Formula (2.1) does not make sense unless $\hat{a}(\omega)_M$ are measurable functions with respect to $(\Omega, \mathbf{P})$. This means that for every Borel set $B\subseteq\mathbb{R}$:

$$\{\omega:\hat{a}(\omega)_M\in B\}\in\mathbf{P}.$$

Such an element of $\mathbf{P}$ is the *subquantum event* that corresponds to the quantum event $\hat{E}^{\hat{a}}(B)$.

Next, we require the following *no-redundancy condition*:

(iii) If $\omega_1, \omega_2\in\Omega$ are such that $\hat{a}(\omega_1)_M=\hat{a}(\omega_2)_M$ for each $\hat{a}\in O(H)$ and each context M containing $\hat{a}$, then $\omega_1=\omega_2$.

(iv) Let $\hat{b}=f(\hat{a})$, where $f:R\to R$ is a continuous function. Then our *final requirement* is to preserve the functional dependence of subquantum values

$$\hat{b}(\omega)_M=f[\hat{a}(\omega)_M] \tag{2.2}$$

for each context M containing $\hat{a}$ and each subquantum state ω.

To specify a *class of simple models* we make the following definition of contexts. We restrict ourselves to contexts of the form $M=A$, where A is the minimal C^* algebra in $L(H)$ generated by an arbitrary observable $\hat{a}$. Then the number $\hat{a}(\omega)_A$ is naturally interpreted as the result of a *minimal* measurement of $\hat{a}$ [15]. By this one means a measurement that gives the exact result of no other observable than a function of $\hat{a}$. In other words, a minimal measurement of $\hat{a}$ also measures all coarser-grained observables, but none of the finer-grained ones.

Suppose that $\hat{a}, \hat{b} \in O(H)$ are related, so that there exist continuous functions $f, g: \mathbb{R} \to \mathbb{R}$ with the properties

$$f(\hat{a})=\hat{b} \qquad \text{and} \qquad g(\hat{b})=\hat{a}. \tag{2.3}$$

Then and only then do $\hat{a}$ and $\hat{b}$ *determine the same minimal context* A. Then, for each $\omega \in \Omega$, we have

$$f[\hat{a}(\omega)_A]=\hat{b}(\omega)_A \qquad \text{and} \qquad g[\hat{b}(\omega)_A]=\hat{a}(\omega)_A. \tag{2.4}$$

One can naturally interpret pairs of observables satisfying (2.3) in the following way: such observables are representing the same measuring procedure, only with different scales. Relation (2.4) actually says that the subquantum theory is invariant under change of scale.

We say that two observables are *equivalent* if (2.4) holds for them. It is easy to see that this determines an equivalence relation in $O(H)$. We shall denote by K the *set of equivalence classes*.

Any other context M that contains $\hat{a}$ is defined analogously by the minimal measurement of another observable $\hat{b}$, such that $\hat{a}=f(\hat{b})$, where f is a *singular* continuous function on the spectrum of $\hat{b}$. The measurement of $\hat{a}$ as a consequence of a minimal measurement of $\hat{b}$ is not minimal for $\hat{a}$, because it contains an excess of information due to the singularity of f. One refers to this measurement of $\hat{a}$ as its *overmeasurement*.

Now the following important question arises: does there exist any subquantum theory satisfying all the above requirements?

There exists at least one realization: the *spectral* one. We shall now construct it. Let us choose from each class $k \in K$ a representative observable $\hat{a}_k \in k$ and *define* the subquantum space Ω to be the Cartesian product

$$\Omega=\prod_{k \in K} \sigma(\hat{a}_k) \tag{2.5}$$

of the *spectra* of the observables $\hat{a}_k$. For each $\hat{a} \in O(H)$ and $\omega \in \Omega$ and every context $A \ni \hat{a}$ we, by definition, take the value

$$\hat{a}(\omega)_A=f[\pi_k(\omega)] \tag{2.6}$$

where $k \in K$ is the unique class such that $\hat{a} \in k$, π_k is a kth coordinate projection and $f: R \to R$ a continuous function satisfying $f(\hat{a}_k)=\hat{a}$. In spite of the non-uniqueness of f, our definition is correct, because, as known in functional analysis, f is unique on $\sigma(\hat{a}_k)$.

In this way we obtain a structure with properties (i), (iii) and (iv). As to property (ii), we first define a σ-field $\mathbf{P}$ on Ω to be the σ-field generated by all inverse images $\pi_k^{-1}(\Lambda)$, where $\Lambda \in B_k$ and B_k is the σ-field of Borel sets in $\sigma(\hat{a}_k)$. For each statistical operator ρ in H we consider the probability measure $\mu_{\rho,k}$ on B_k defined by the equality

$$\mu_{\rho,k}(\Lambda)=\mathrm{Tr}(\rho \hat{P}_\Lambda(\hat{a}_k))$$

$\Lambda \in B_k$, and $\hat{P}_\Lambda(\hat{a}_k)$ is the spectral measure of Λ associated with $\hat{a}_k$.

For any measurable essentially bounded function (with respect to the spectral measure) $h:\sigma(\hat{a}_k)\to C$ one has

$$h(\hat{a}_k)=\int_{\sigma(\hat{a}_k)} h(\lambda)\,\mathrm{d}P_\lambda(\hat{a}_k)$$

and, accordingly,

$$\mathrm{Tr}(\rho h(\hat{a}_k))=\int_{\sigma(\hat{a}_k)} h(\lambda)\,\mathrm{d}\mu_{\rho,k}(\lambda).$$

Finally, we define a probability measure μ_ρ on **P** to be the direct product of measures $\mu_{\rho,k}$. It is then easy to see that the equality $\mathrm{Tr}(\rho\hat{a})=\int_\Omega \hat{a}(\omega)_A\,\mathrm{d}\mu_\rho(\omega)$ of requirement (ii) holds because on the right-hand side the integral factorizes and all factors are equal to one, except the kth.

We shall now discuss *contextual features* of the spectral model, constructed above. Contextuality consists in the fact that one and the same observable generally has different values if the system is in the same subquantum state but if the observable is measured in different ways.

To elaborate on this, let us suppose that a measurement of an observable $\hat{a}\in O(H)$ is performed. This measurement is *ipso facto* a measurement of any observable $\hat{b}\in O(H)$ of the form $\hat{b}=f(\hat{a})$, for f continuous, i.e. for $\hat{b}$ which belongs to the context A determined by $\hat{a}$. In this case the result of the measurement of $\hat{b}$ is $f(\hat{a}(\omega)_A)$, where $\omega\in\Omega$. However, if f is singular on $\sigma(\hat{a})$ then $\hat{b}$ also belongs to other contexts, e.g. to B determined by $\hat{b}$ itself—corresponding to the minimal measurement of $\hat{b}$. In this case the result is $\hat{b}(\omega)_B$, and, as a rule, $\hat{b}(\omega)_B\neq f(\hat{a}(\omega)_A)$. Hence, we are dealing with a contextual subquantum model.

We shall illustrate the spectral realization of a simple contextual subquantum model in a finite-dimensional Hilbert space, in which all observables have discrete spectra. In other words, each Hermitian operator can uniquely be written in the spectral form

$$\hat{a}=\sum_n a_n\hat{p}_n \tag{2.7a}$$

where all characteristic values a_n are distinct. The corresponding characteristic decomposition of the identity operator is

$$\hat{\imath}=\sum_n \hat{p}_n. \tag{2.7b}$$

The concept of the *minimal measurement* of $\hat{a}$ is defined [15] by requiring: (i) that a physical system for which the measurement of $\hat{a}$ gave a_n has this characteristic value with certainty in the case of an immediate repetition of the same measurement; (ii) that the change of the quantum state (statistical operator) ρ in the measurement be minimal (in the sense of the distance in the operator Hilbert space). Under these requirements it was shown [15] that the selective measurement of a_n of $\hat{a}$ converts ρ into

$$(\mathrm{Tr}\,\rho\hat{p}_n)^{-1}\hat{p}_n\rho\hat{p}_n. \tag{2.8}$$

However, there also exists the possibility of *overmeasurement* of observable $\hat{a}$ consisting of the minimal measurement of another observable $\hat{b}$, the spectral form of which is

$$\hat{b}=\sum_n \sum_{k_n} b_{n,k_n}\hat{p}_{n,k_n} \tag{2.9}$$

where all b_{n,k_n} are distinct. We are dealing here with a non-trivial continuation of decomposition of the characteristic projectors $\hat{p}_n$ of the observable $\hat{a}$, which proceeds as follows:

$$\forall n: \hat{p}_n=\sum_{k_n} \hat{p}_{n,k_n} \tag{2.10}$$

where at least for one value of n there are at least two terms in (2.10).

Equivalently, one can say that $\hat{a}$ is a function of $\hat{b}$, i.e. $\hat{a}=f(\hat{b})$, where 'f' maps the spectrum of $\hat{b}$ on to that of $\hat{a}$ as follows:

$$\forall n, k_n: f(b_{n,k_n})=a_n. \tag{2.11}$$

Evidently, the map 'f' is *singular* (or non-injective) on the spectrum of $\hat{b}$.

From the physical point of view, overmeasurement is a consequence of the simultaneous measurement of $\hat{a}$ and another observable $\hat{c}$ that is compatible (commutes) with $\hat{a}$, but $\hat{c}$ has a characteristic decomposition of the identity operator distinct from (2.7*b*). Each characteristic subspace of $\hat{a}$, i.e. $R(\hat{p}_n)$, is invariant under $\hat{c}$, and its reducee, when spectrally decomposed, breaks up $R(\hat{p}_n)$:

$$R(\hat{p}_n)=\sum_{k_n}^{\oplus} R(\hat{p}_{n,k_n}) \tag{2.10'}$$

where the subspaces $R(\hat{p}_{n,k_n})$ are the intersections of $R(\hat{p}_n)$ and those characteristic subspaces of $\hat{c}$ that do intersect non-trivially with $R(\hat{p}_n)$.

Thus, the purpose of introducing the above observable $\hat{b}$ is to replace formally the *pairs of characteristic values of $\hat{a}$ and $\hat{c}$ with the individual characteristic values b_{n,k_n}* of $\hat{b}$.

In order to introduce contexts in a simple way, one can break up the set of all observables $O(H)$ into equivalence classes corresponding to the equivalence relation (2.3), which actually means that $\hat{a}$ and $\hat{b}$ are functions of each other, non-singular on the spectra. One should note that in a finite-dimensional state space all functions are continuous (in the discrete topology). We denote the quotient set by K.

In each of these classes are those observables that can be simultaneously minimally measured. All observables belonging to one and the same class are characterized by a common characteristic decomposition of the identity operator.

Since the minimal measurement of an observable $\hat{a}$ is at the same time also overmeasurement of all observables that are its functions $f(\hat{a})$ singular on the spectrum of $\hat{a}$, the *minimal context* A of $\hat{a}$ consists of all above equivalence classes to which $\hat{a}$ and all $f(\hat{a})$ belong.

The suggested model of the spectral realization is based on the idea that for an individual quantum system all observables belonging to a given context (the above-described family of classes from K) have a definite value. Since the contexts overlap, one and the same observable may have different values for the same individual system, depending on the context, i.e. on the choice of the minimally measured observable.

By definition, contexts appear due to the possibility of overmeasurement, i.e. due to degenerate characteristic values. In the case of the two-dimensional state space, there are no non-trivial observables with degenerate characteristic values. Hence, in this case each observable belongs to only one context, and, consequently, contexts are not necessary [4].

In state spaces of three and more dimensions there are infinitely many possibilities of overmeasurement of observables that have degenerate characteristic values. The question then arises of which overmeasurements are equivalent. Evidently, this is the case when $\hat{a}=f(\hat{b})$ and $\hat{a}=g(\hat{b}')$, and $\hat{b}$ and $\hat{b}'$ have the same characteristic decomposition of the identity.

Thus the finite-dimensional model gives a simple illustration of the basic and plausible idea that a subquantum theory, i.e. individual-system description, is possible only if one takes into account the complete measurement procedure (here described by contexts).

3. C^*-algebraic formulation of the contextual subquantum model

It is obvious from the preceding section that the concept of context is the price one has to pay for the very existence of a causal subquantum model. At first glance, contextuality appears as an unnatural notion among the basic ones like states and variables. However, it can be shown [16] that a C^*-algebraic formulation of the subquantum model can be made free of contextuality by an appropriate extension of the C^* algebra of quantum observables into a C^* algebra of subquantum variables. However, the purpose of this article is to view the quantum states as context-independent distributions in the subquantum space, and for this task the concept of context seems unavoidable. To achieve this aim, the C^*-algebraic approach is mathematically best suited. Now we turn to the C^*-algebraic formulation in $L(H)$.

We start by reformulating the set of all conditions of the preceding section in the C^*-algebraic language. First of all, it is easy to see that relation (2.3) between two observables $\hat{a}, \hat{b} \in O(H)$ holds if and only if they generate the same commutative C^* subalgebra of $L(H)$. Classes of equivalent observables are in a natural bijection with the commutative C^* subalgebras of $L(H)$ that are generated by a single bounded observable.

Such a subalgebra A consists precisely of all continuous functions of the generating observable. We take it as a context.

Secondly, assumptions (i) and (iv) can be reformulated as follows:

*For each subquantum state $\omega \in \Omega$, each context A and $\hat{a} \in A$ the number $\hat{a}(\omega)_A \in C$ (the value of $\hat{a}$ in ω, relative to A) is defined. The map $A \ni \hat{a} \to \hat{a}(\omega)_A$ is a non-trivial multiplicative linear *-functional, i.e. a character on A.*

In any given subquantum space Ω, *a topology* can be introduced in a natural way: for each context A we require all functions $\Omega \ni \omega \to \hat{a}(\omega)_A$ to be continuous, and the topology to be minimal in this sense. In this induced topology the subquantum space need not be compact. However, it has been shown [17] that it can always be compactified.

An algebraic formulation which also encompasses (ii) and (iii) reads as follows:

*The subquantum space Ω is a compact topological space. For each context A a *-homomorphism $F_A : A \to C(\Omega)$ is defined, where $C(\Omega)$ denotes the C^* algebra of complex-valued continuous functions on Ω. The number $F_A(\hat{a})(\omega)=\hat{a}(\omega)_A$ is the value of $\hat{a}$ in the state ω relative to the context A. For each statistical operator ρ in H there exists* *a probability measure μ_ρ on the Borel σ-field $B(\Omega)$ of Ω such that for each context A and $\hat{a}\in A$ the following equality holds:*

$$\mathrm{Tr}(\rho\hat{a})=\int_\Omega F_A(\hat{a})(\omega)\,\mathrm{d}\mu_\rho(\omega).$$

If $F_A(\hat{a})(\omega_1)=F_A(\hat{a})(\omega_2)$ for each A and $\hat{a}\in A$ then $\omega_1=\omega_2$.

4. An abstract C^*-algebraic formulation of a general contextual subquantum model and its structure space realization

In the preceding section we formulated a subquantum model in terms of a C^* algebra, the elements of which have a particular nature: they are operators in the quantum state space H. So far, we have considered contexts which are commutative C^* subalgebras generated by one of its Hermitian elements. In view of the fact that in quantum mechanics one typically has sets of compatible observables that are simultaneously measured, we extend our definition of context to commutative C^* algebras of a general nature. We find it more convenient to work in terms of an abstract C^* algebra.

Let Σ be a C^* algebra and $\mathbf{T}$ a chosen family of commutative C^* subalgebra of Σ that generates the whole Σ (a minimality requirement for Σ).

Definition 4.1. A *contextual subquantum model* for given $(\Sigma,\mathbf{T})$ is a pair $(\Omega,\{F_A;A\in\mathbf{T}\})$, where Ω is a compact topological space and $\{F_A;A\in\mathbf{T}\}$ is a family of *-homomorphisms $F_A:A\to C(\Omega)$ such that:

(i) For each state ρ on Σ there exists a probability measure μ_ρ on $B(\Omega)$ satisfying

$$\int_\Omega F_A(a)(\omega)\,\mathrm{d}\mu_\rho(\omega)=\rho(a)$$

for each $A\in\mathbf{T}$ and $a\in A$.

(ii) If $F_A(a)(\omega_1)=F_A(a)(\omega_2)$ for each $A\in\mathbf{T}$ and each $a\in A$, then $\omega_1=\omega_2$.

Remark. In the special case $\Sigma=L(H)$ the states ρ which are ultraweakly continuous on $L(H)$ are precisely those which are representable by statistical operators in H.

Theorem 4.1. For any pair $(\Sigma,\mathbf{T})$ there exists a contextual subquantum model: *the structure space realization.*

Proof. Let us consider the Cartesian product of the structure spaces (see appendix 2)

$$\bar{\Omega}=\prod_{A\in\mathbf{T}}X(A)$$

endowed with the Tihonov product topology [18]. In this topology, $\bar{\Omega}$ is compact. For each $A\in\mathbf{T}$ we define $\bar{F}_A:A\to C(\Omega)$ as follows:

$$\bar{F}_A(a)(\omega)=\pi_A(\omega)(a)$$

where $\pi_A:\bar{\Omega}\to X(A)$ denotes the Ath coordinate projection. It is clear that $\bar{F}_A$ is a *-homomorphism and that property (ii) of definition 4.1 is fulfilled. For each state ρ on Σ and $A\in\mathbf{T}$, let $\mu_{\rho,A}$ denote the probability measure on the Borel σ-field of $X(A)$, corresponding (via the Riesz theorem) to the restriction $\rho|A$. Finally, we define a probability measure μ_ρ on $\bar{\Omega}$ to be the direct product of all measures $\mu_{\rho,A}$. □

Our next theorem relates the *structure space subquantum model* $(\bar{\Omega}, \{\bar{F}_A; A\in\mathbf{T}\})$ with an arbitrary subquantum model satisfying definition 4.1. The subquantum space $\bar{\Omega}$ contains, in a natural manner, the subquantum space Ω of any other such model.

Theorem 4.2. Let $(\Omega, \{F_A; A\in\mathbf{T}\})$ be an arbitrary contextual subquantum model for $(\Sigma, \mathbf{T})$. Then there exists one and only one map $\iota: \Omega \to \bar{\Omega}$ such that $\lambda_A = \pi_A \iota$, for each $A\in\mathbf{T}$, where the map $\lambda_A: \Omega \to X(A)$ is defined by

$$\lambda_A(\omega)(a) = F_A(a)(\omega).$$

The map ι is injective, it is a homeomorphism of Ω on to $\iota(\Omega)$, and it satisfies $F_A(a) = \bar{F}_A(a)\iota$, for each $A\in\mathbf{T}$ and $a\in A$. Hence, the entire model $(\Omega, \{F_A; A\in\mathbf{T}\})$ is embedded into the model $(\bar{\Omega}, \{\bar{F}_A; A\in\mathbf{T}\})$, making the latter *maximal.*

Proof. The existence and uniqueness of ι is a direct consequence of the definition of the space $\bar{\Omega}$. As is well known, a necessary and sufficient condition for continuity of a map $f: Z \to \bar{\Omega}$ where Z is an arbitrary topological space, is the continuity of all the compositions $\pi_A f: Z \to X(A)$. Consequently, ι is continuous, because the λ_As have this property. (The continuity of λ_A is a consequence of the fact that the topology on $X(A)$ is defined as the *-weak topology.) The relation $\lambda_A = \pi_A\iota$ is actually $\lambda_A(\omega)(a) = F_A(a)(\omega) = [\pi_A\iota(\omega)](a) = [\bar{F}_A(a)\iota](\omega)$. Thus, $F_A(a) = \bar{F}_A(a)\iota$.

Further, $\iota(\omega_1) = \iota(\omega_2)$ implies $F_A(a)(\omega_1) = F_A(a)(\omega_2)$, for each $A\in\mathbf{T}$ and $a\in A$. Thus $\omega_1 = \omega_2$, that is, ι is injective.

Finally, ι as a continuous bijection between the compact topological space Ω and the Hausdorff topological space $\iota(\Omega)$ is necessarily a homeomorphism [18]. □

The existence of non-maximal models is discussed in appendix 3. The following proposition shows that the subquantum space of any contextual subquantum model is sufficiently large in a certain sense.

Proposition 4.3. Let $(\Omega, \{F_A; A\in\mathbf{T}\})$ be a contextual subquantum model for $(\Sigma, \mathbf{T})$. For each $A\in\mathbf{T}$ one has $\pi_A\iota(\Omega) = X(A)$, i.e. after embedding each projection is the entire $X(A)$. Equivalently, for each $A\in\mathbf{T}$, the *-homomorphism F is injective.

Proof. If $F_A(a) = 0$ then for each state ρ on Σ we have

$$\rho(a) = \int_\Omega F_A(a)(\omega)\, \mathrm{d}\mu_\rho(\omega) = 0.$$

Thus $a = 0$. □

Remark. The structure space realization specifies to the spectral one if $\Sigma = L(H)$ and the contexts are defined as in section 2 (see appendix 2).

5. Discussion

(*a*) It is well known that mathematically the most natural way of representing a general observable is via its spectral form, in which the spectrum and the spectral projectors are treated with equal physical significance. In quantum mechanics, there correspond characteristic vectors to the discrete spectral points, which have the physical meaning of homogeneous ensembles with sharp values of the observable. In contrast, there are no characteristic vectors corresponding to the points of the continuous part of the spectrum, i.e. these values cannot be sharp in any ensemble.

On the other hand, on the subquantum level both the points of the discrete part of the spectrum and those of the continuous one are treated on the same footing: in a subquantum state $\omega \in \Omega$ each quantum observable (for a given context) has a definite value regardless of the nature of this value as a spectral point. Thus, in this article we have expounded a consistent subquantum (contextual) theory covering all bounded observables (with any kind of spectra).

If the general theory (see section 4) is specified to the case when contexts are defined as commutative von Neumann algebras, then along with a given observable all its spectral projectors are explicitly interpretable as subquantum events. As mentioned in the introduction, the price one has to pay for this is that the subquantum space Ω is necessarily extremely disconnected.

On the other hand, if the general theory is specified to the case when contexts are defined as commutative C^* algebras generated by single observables, not all spectral projectors of a given observable appear as subquantum events.

At first glance one becomes worried about the measurement of the points of the continuous spectra because this measurement is necessarily approximate and involves the occurrence of spectral projectors of intervals containing the points. Though these are generally not present among the subquantum events, they can be arbitrarily well approximated by appropriate continuous functions of the generating observable (which are, admittedly, more complex observables).

(*b*) The approach of this article is based on the concept of contextuality. However, this concept is not an accepted basic physical one like variables, states and averages. A natural way to view contextuality is to define subquantum variables as ordered pairs (a, A). A theory of such subquantum variables was elaborated in the framework of *contextual extensions* of C^* algebras of quantum observables [16, 24].

(*c*) The most intriguing aspect of contextuality in the possibility of non-locality on the subquantum level. More precisely, in the case of two particles which are distant but in a statistically correlated quantum state the *choices* that one decides to measure on one of them represent different contexts for a simultaneous measurement on the other. Actually, the phenomenon of non-local contextuality is closely related to the kind of probability theory on which the subquantum theory is based. In standard probability theory, which is used in this article, non-locality is unavoidable, as is well known from Bell's inequalities [10, 19]. On the other hand, it was shown [20–23] that with a generalized probability theory it is possible to construct a *local* subquantum theory.

(*d*) It is clear in the theory presented in this article that for a complete observable there exists only one context: the one generated by it. Occasionally in the literature on hidden variables [12] one encounters theories that are practically restricted to complete observables. This is why contextuality is not conspicuous in them. One should bear in mind that every complete observable for a given quantum system becomes incomplete if one considers a more complex system containing the former system as a subsystem. Hence, restriction to complete observables is an oversimplification that amounts to hiding contextuality.

(*e*) When a subquantum model is contextual, this does not necessarily imply that every incomplete observable is contextual. If one or more are non-contextual, then one could construct an intermediate or partially causal theory in which quantum mechanics is completed by subquantum states in which only the non-contextual observables have (unique) definite values. In this way one obtains a theory of the type of beables (as Bell called them [10]). In Bell's article the densities of the numbers of fermions in space and time were taken for beables.

Appendix 1. Gleason's theorem and the impossibility of non-contextual subquantum states

We are going to prove that Gleason's theorem [3] forbids the existence of non-contextual subquantum states, as far as standard quantum mechanical structure is concerned. Let H be an (infinite-dimensional) Hilbert space. Further, let $\mathbf{T}$ be a collection of commutative C^* subalgebras (contexts) of $L(H)$ which is sufficiently rich in the sense that for each pair $(\hat{p}, \hat{q})$ of mutually orthogonal projectors in H there exists $A \in \mathbf{T}$ such that $\hat{p}, \hat{q} \in A$. In particular, this condition implies that every projector is contained in some context $A \in \mathbf{T}$.

Let $(\Omega, \{F_A; A \in \mathbf{T}\})$ be an arbitrary contextual subquantum model for $(L(H), \mathbf{T})$.

Theorem A1. For each subquantum state $\omega \in \Omega$ there exists a projector $\hat{p} \in P(H)$ and contexts $A, B \in \mathbf{T}$ such that $\hat{p} \in A \cap B$ and

$$F_A(\hat{p})(\omega) = 1$$

$$F_B(\hat{p})(\omega) = 0.$$

In other words, every subquantum state $\omega \in \Omega$ is necessarily *contextual.*

Proof. It is easy to see that $F_A(\hat{p})(\omega) \in \{0, 1\}$ for each projector $\hat{p} \in P(H)$, $\omega \in \Omega$ and context $A \in \mathbf{T}$ such that $\hat{p} \in A$. Let us assume that there exists a subquantum state $\omega \in \Omega$ with the property that the number $F_A(\hat{p})(\omega)$ is independent of A for a given $\hat{p}$. Hence, the formula $\psi(\hat{p}) = F_A(\hat{p})(\omega)$ consistently defines a map $\psi: P(H) \to \{0, 1\}$.

For a given $n \geqslant 3$ let us consider an arbitrary unital *-monomorphism $i: L(H_n) \to L(H)$, where H_n is an n-dimensional Hilbert space. For example, such a map can be constructed by realizing H in the form $H \cong H_n \otimes H$ and defining $i(a) = a \otimes I$.

The composition $\psi i: P(H_n) \to \{0, 1\}$ is a normalized and additive map (in the sense that $\psi i(\hat{p} + \hat{q}) = \psi i(\hat{p}) + \psi i(\hat{q})$ if the projectors $\hat{p}, \hat{q}$ are orthogonal). According to Gleason's theorem, there exists a statistical operator $\rho: H_n \to H_n$ such that $\psi i(\hat{p}) = \mathrm{Tr}(\rho \hat{p})$ for each $\hat{p} \in P(H_n)$. However, this is a contradiction, since the set $\{\mathrm{Tr}(\rho \hat{p}); p \in P(H_n)\}$ contains at least one point between 0 and 1. Hence, non-contextual subquantum states do not exist. □

Remark. The above theorem implies that not all continuous functional relations between quantum observables are preserved on the subquantum level unless a context is specified. Moreover, they cannot be preserved in a single subquantum state. Indeed, if ω is a subquantum state respecting all continuous functional relations, and if $\mathbf{T}$ consists of contexts generated by single observables, then

$$\hat{a}(\omega)_A = \hat{a}(\omega)_B$$

for each $B \in \mathbf{T}$ and $\hat{a} \in B$, where A is the context generated by $\hat{a}$. Here, we have used the notation of section 2. In particular, the above formula holds for $\hat{a} \in P(H)$. However, this is in contradiction to theorem A1.

In the above proof, Gleason's theorem is not directly applicable on the projector lattice $P(H)$, since in the infinite-dimensional case an additional assumption of *countable additivity* figures in the formulation of the theorem. It is worth noticing that the contradiction emerges without any assumption about statistical interpretability of quantum states.

It is interesting to analyse a possibility of non-contextual subquantum states in the general C^*-algebraic framework. There exist various highly non-commutative C^* algebras Σ 'admitting' non-contextual subquantum states. For example, if algebra possesses characters, then every character ($\Leftrightarrow$ dispersion-free state) is interpretable in the mentioned manner. On the other hand, it is important to mention that the algebraic version of Gleason's theorem [25] excludes the possibility of non-contextual subquantum states for the large class of von Neumann algebras Σ without direct summands of the type I_n $n \in \{1, 2\}$ (if the family $\mathbf{T}$ is sufficiently large, as is assumed in the above consideration).

Appendix 2. The structure space of a commutative C^* algebra

Let A be a commutative C^* algebra and let $X = X(A)$ be the set of all characters on A (i.e. the set of all non-trivial linear multiplicative functionals on A). It is worth noticing that every character $f: A \to C$ is a continuous Hermitian map satisfying $f(1) = 1$. Let A^* be the space of all continuous linear functionals on A. In the *-weak topology of A^* (this topology is generated by the system of seminorms $p_a: f \to |f(a)|$, where $a \in A$) the set X is a compact topological space, and it is called the *structure space* of A (or the spectrum of A).

Let us now consider a commutative C^* algebra $C(X)$ of all continuous complex-valued functions on X.

According to the Gelfand–Naimark theorem [14], the mapping $G: A \to C(X)$ defined by

$$G(a)(f) = f(a)$$

where $a \in A$ and $f \in X$ is a C^*-algebraic isomorphism.

Topological properties of X are encoded in algebraic properties of A. For example, metrizability of X is equivalent to separability of A.

In the case when A is generated by a single element $a \in A$, then $X(A)$ is naturally homeomorphic to the standard spectrum $\sigma(a)$. In terms of the identification $X(A) = \sigma(a)$, we have $G(a)(\omega) = \omega$, for each $\omega \in \sigma(a)$.

Appendix 3. On the existence of non-maximal subquantum models

According to theorem 4.2, every contextual subquantum model $(\Omega, \{F_A; A \in \mathbf{T}\})$ for $(\Sigma, \mathbf{T})$ is naturally embedable in the maximal model $(\bar{\Omega}, \{\bar{F}_A; A \in \mathbf{T}\})$.

If the algebra Σ and the family $\mathbf{T}$ are chosen in a 'sufficiently singular' way, then the maximal model will be the unique contextual subquantum model. As a trivial illustration for this, we can take any pair $(A, \{A\})$, where A is a commutative C^* algebra.

However, in the general case there exists a large variety of non-maximal models. We are going to describe a simple construction of such models.

Let us assume that there exists a non-trivial map $\varphi: \mathbf{T} \to \mathbf{T}$ such that $A \subseteq \varphi(A)$. Let us define the space Ω and the family $\{F_A; A \in \mathbf{T}\}$ of maps $F_A: A \to C(\Omega)$ as follows:

$$\Omega = \prod_{A \in \varphi(\mathbf{T})} X(A)$$

$$F_A(a)(\omega) = \pi_{\varphi(A)}(\omega)(a)$$

where πs are the corresponding coordinate projections. It is easy to see that the pair $(\Omega, \{F_A; A \in \mathbf{T}\})$ is a contextual subquantum model for $(\Sigma, \mathbf{T})$ and that the image of the map $\iota: \Omega \to \bar{\Omega}$ figuring in theorem 4.2 is a non-trivial subspace of $\bar{\Omega}$.

It is of some interest to define a concept of a minimal subquantum model. By definition, a contextual subquantum model is *minimal* if it does not contain a subquantum space of another subquantum model as its non-trivial part. It can be shown [17] that every contextual subquantum model has a minimal submodel.

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